Physics 716 Final problem set Spring 2018

Due May 3, 2018

Problems

1. *Background field method and QCD β function.* Recall that the QCD β function is determined by

$$\beta(g) = \lim_{\epsilon\to 0} \frac{dg}{d\log\mu} = g^2 \frac{\partial Z_{g,1}}{\partial g}\,, \tag{1}$$

where $g(\mu)$ is the renormalized coupling and Z_g is the renormalization constant using dimensional regularization, $d = 4 - 2\epsilon$, and the $\overline{\rm MS}$ scheme,

$$g_{\rm bare} = Z_g \mu^\epsilon g(\mu)\,, \quad Z_g = 1 + \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} Z_{g,n}\,. \tag{2}$$

a) In class we computed the contribution to the β function from a Dirac fermion ψ transforming in a general representation of the gauge group:

$$\Delta\beta_\psi = \frac{g^3}{(4\pi)^2}\frac{4}{3} C(r_\psi)\,, \tag{3}$$

where $C(r)$ is the normalization constant for representation r. Compute the contribution from a hypothetical complex scalar field that transforms in the fundamental of the gauge group SU(3).

b) Same as part (a), but for a real scalar in the adjoint representation of SU(3).

c) In class we derived the gauge fixed action using the background field method, i.e., splitting the guage field into background (low frequency) and quantum (high frequency) components

$$A(x) = A_B(x) + A_Q(x)\,, \tag{4}$$

and applying the Fadeev Popov procedure with gauge fixing function for the path integral over A_Q:

$$f^a(A_Q) = \partial_\mu A_Q^{a\,\mu} + g f^{abc} A^b_{B\,\mu} A_Q^{c\,\mu}\,. \tag{5}$$

For a general gauge-fixing parameter ξ, derive the Feynman rules for three point vertices $A_Q A_Q A_B$ and $c\bar{c}A_B$, where c is the ghost field.

d) Using the Feynman rules from part (c) and a general choice of gauge fixing parameter ξ, compute the β function through one loop order for a pure SU(n) gauge theory (you may specialize to $n = 3$).

e) For the case of SU(3) gauge theory, what is the restriction on the number of colored Dirac fermions in order that the theory is asymptotically free?

2. *Strong CP problem and axion model.* In the Standard Model, quark masses arise from the Yukawa interactions involving the Higgs field and quark bilinears,

$$\mathcal{L}_{\rm Yukawa} = -\bar{u}_R \lambda_u H^\dagger Q_L - \bar{d}_R \lambda_d \tilde{H}^\dagger Q_L + \text{h.c.}\,. \tag{6}$$

Here λ_u and λ_d are $n_G \times n_G$ matrices acting on the n_G component fields u_R, d_R and Q_L, where n_G is the number of generations.

a) Decompose the complex matrices λ_u and λ_d as

$$\lambda_u = R_u \hat{\lambda}_u L_u^\dagger\,, \quad \lambda_d = R_d \hat{\lambda}_d L_d^\dagger\,, \tag{7}$$

where $\hat{\lambda}_u$ and $\hat{\lambda}_d$ are real diagonal matrices. Notice the phase ambiguity in the choice of R_u and L_u:

$$R'_u = R_u \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_{n_G}}), \quad L'_u = L_u \mathrm{diag}(e^{i\alpha_1}, \cdots, e^{i\alpha_{n_G}}), \tag{8}$$

and similarly for R_d and L_d. Now consider the quark mass eigenstate basis (denoted with primes)

$$u_R = R_u u'_R, \quad d_R = R_d d'_R, \quad u_L = L_u u'_L, \quad d_L = L_d d'_L. \tag{9}$$

Show that all interactions become flavor diagonal in the mass eigenstate basis, with the exception of the charged-current weak interactions involving $W^\pm_\mu$.

b) Define the CKM matrix appearing in charged-current weak interactions as

$$V_{\rm CKM} = L_u^\dagger L_d. \tag{10}$$

Taking into account the phase redundancy in part (a), how many independent parameters are present in $V_{\rm CKM}$ for $n_G = 1$, $n_G = 2$ and $n_G = 3$?

c) Consider adding to the standard model the renormalizable, gauge-invariant interaction,

$$\mathcal{L}_\theta = -\theta \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{11}$$

Under an axial transformation of a single quark,

$$q \to e^{-i\phi\gamma_5} q, \tag{12}$$

the Jacobian of the path integral measure corresponds to the anomalous shift

$$\mathcal{L} \to -\phi \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}. \tag{13}$$

Generalize to the complete transformation that diagonalizes the quark mass matrix, and show that in the mass eigenstate quark basis, the θ term becomes

$$\theta' = \theta - \arg\det(M_{\rm quark}), \tag{14}$$

where $M_{\rm quark}$ is the mass matrix in the original, weak interaction, basis.

d) In Nature, it is found that θ' is extremely small, $\sim 10^{-9}$, the result of a cancellation between apparently unrelated parameters (θ, and the parameters in $M_{\rm quark}$). This is the so-called "strong CP problem". As discussed in class, if θ' were replaced by an arbitrary field $\theta'(x)$, the lowest vacuum energy of the system is achieved for $\theta' = 0$. Consider an extension of the Standard Model by a new "quark" ψ [i.e., ψ transforms in the fundamental of color SU(3)] and a new gauge singlet complex scalar field σ:

$$\mathcal{L}_{\rm new} = |\partial_\mu \sigma|^2 + \bar{\psi} i \not{D} \psi - \kappa \left(\sigma \bar{\psi}_L \psi_R + \mathrm{H.c.} \right) - V(\sigma), \tag{15}$$

Suppose that σ acquires a vacuum expectation value. Show that ψ then acquires a mass, and assume this mass is large compared to any mass scale in the Standard Model. Consider the substitution $\sigma(x) = [f + r(x)] e^{ia(x)/f}$. Integrate out the fermion ψ and the radial mode r to show that at low energy, the effects of the ψ and σ appear as a new effective interaction involving the massless axion $a(x)$,

$$\mathcal{L}_{\rm axion} = \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{64\pi^2} \frac{a(x)}{f} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}, \tag{16}$$

What are the implications for the strong CP problem?