For systems involving the NGBs of spontaneously broken symmetries, effective theories are constructed from fields living on the coset space G/H, where G is the full symmetry group and H is the unbroken subgroup. In several important cases, the relevant G and H are such that the coset space has nontrivial topology and allows a class of interactions for which the full G symmetry is not manifest.

1+1 dimensional example

Consider a $1+1$ dimensional field theory for the NGB's of G $= O(4)$ to H $= O(3)$. This symmetry breaking pattern corresponds to a VEV for a complex $SU(2)$ doublet:

$$H = \begin{pmatrix} H_1 + iH_2 \\ H_3 + iH_4 \end{pmatrix}, \quad \langle H \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1}$$

The coset space G/H is topologically identical to the three-sphere S^3, and fields Φ living on this space may be written as

$$\Phi(x) = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \quad \Phi^\dagger \Phi = \sum_{i=1}^{4} (\phi_i)^2 = 1\,. \tag{2}$$

We look for G-invariant interactions. At two-derivative order,

$$\mathcal{L} = c_1 \partial_\mu \Phi^\dagger \partial^\mu \Phi + c_2 \Phi^\dagger \partial_\mu \Phi \Phi^\dagger \partial^\mu \Phi + \dots\,. \tag{3}$$

In fact, another G-invariant interaction may be written at two-derivative order. We will first construct this interaction "out of thin air", then describe the more general existence of such terms. We will also discuss the necessity of such terms and an alternative manner of deriving them in theories with underlying fermions.

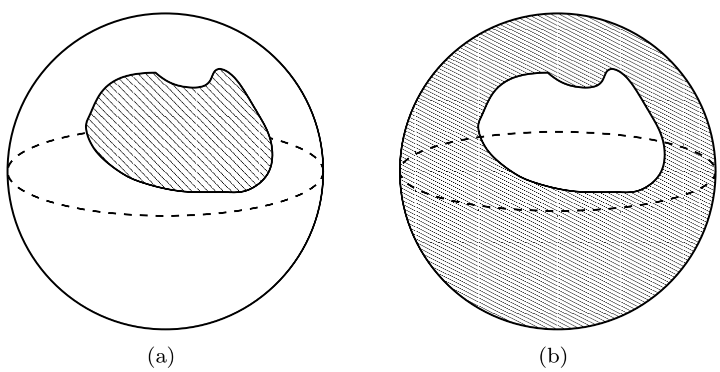

Figure 1: The action corresponds to the area bounded by the image of spacetime. Two different bounding surfaces are pictured.

The new term is constructed in a coordinate space with periodic time coordinate, such that spactime may be identified with the two-sphere S^2. The fields thus represent a mapping from S^2 into S^3, i.e., $x \to \Phi(x)$. Such a mapping is illustrated in Fig. 1. The new term in the action is simply defined as the area contained by the image of spacetime under this mapping. This area is clearly invariant under the full $G = O(4)$ group of rotations.

To express this action in a more conventional lagrangian language, consider the volume element on S^3, which may be written in differential form language[1]

$$\omega = \frac{1}{2} \Phi^\dagger d\Phi d\Phi^\dagger d\Phi\,. \tag{4}$$

[1] For a review of differential forms, see XXX.

Exercise: In the vicinity of $\phi_N = \pm 1$, the volume form on the $(N-1)$-sphere S^{N-1} embeddded in $\mathbb{R}^N$ with $\phi_1^2 + \cdots + \phi_N^2 = 1$ may be expressed as

$$\omega = \frac{1}{\sqrt{1 - \sum_{i=1}^{N} (\phi_i)^2}} d\phi_1 d\phi_2 \cdots d\phi_{N-1} \,. \tag{5}$$

Show that Eq. (4) reduces to this form for $N = 4$.

Exercise: Integrate the result from preceding exercise to obtain the volume of S^n, $n = 1, 2, \ldots, 5$.

The action may thus be written as

$$\Gamma(\Phi) = \text{const.} \times \int_{M^3} \omega \,, \tag{6}$$

where M^3 denotes the three-dimensional region with the image of spacetime as its boundary, depicted as the shaded region in Fig. 1(a). In fact, any given mapping $\Phi(x)$, determined two such regions, and there is no invariant way to decide which to use. To build a consistent theory, we require that the difference in Γ between choices yield a multiple of 2π, so that observables derived from $\exp(i\Gamma)$ are well-defined. It is straigtforward to see that the constant in Eq. (6) must be an integer multiple of (2π) divided by the volume, $2\pi^2$, of S^3, yielding the final result (for integer p),

$$\Gamma(\Phi) = \frac{p}{\pi} \int_{M^3} \omega \,. \tag{7}$$

As a differential form, the volume element is closed, in the sense that

$$d\omega = 0 \,, \tag{8}$$

which follows from the fact that any differential four-form on a three-dimensional manifold must vanish. The volume element is not exact, in the sense that $\omega \neq d\eta$ for any two-form η; this follows from the fact that the integral of ω over the entire S^3 is nonvanishing. However, the fact that $d\omega = 0$ implies that ω is locally exact, in the sense that it can be written as $d\eta$, for some η, in a finite region around any particular point in field space. In particular, expanding at small field values, at each order in the expansion we may write the result as a total derivative. For example, around $\phi^4 = 0$,

$$\begin{aligned} \omega &= \left[1 + \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2) + \ldots \right] d\phi_1 d\phi_2 d\phi_3 \,, \\ &= d \left[\phi_1 d\phi_2 d\phi_3 + \frac{1}{6} \left(\phi_1^3 d\phi_2 d\phi_3 + \phi_2^3 d\phi_3 d\phi_1 + \phi_3^3 d\phi_1 d\phi_2 \right) + \ldots \right] \,. \end{aligned} \tag{9}$$

Using the divergence theorem ($\int_M d\eta = \int_{\text{boundary of } M} \eta$)

$$\Gamma = \int d^2x \, \frac{p}{\pi} \epsilon^{\mu\nu} \phi_1 \partial_\mu \phi_2 \partial_\nu \phi_3 + \ldots \,. \tag{10}$$

This is an interaction term of a $(1+1)$-dimensional Lagrangian that can be treated by the usual methods of perturbation theory, etc.

Generalities

The simple example of the previous section illustrates the main ingredients for topological interactions that can be easily generalized to other symmetry groups, and to four-dimensional spacetime. The essential property of the $G/H = S^3$ example is the existence of a differential three-form ω on the field space that is

- invariant under the full group G
- closed, $d\omega = 0$

- not exact, $\omega \neq d\eta$.

The first property makes G invariance explicit. The second property ensures that the interaction can be expressed, for small fields, as a four dimensional Lagrangian. The third property implies that the interaction is not equivalent to an ordinary, manifestly G-invariant, four-dimensional Lagrangian term.

In four spacetime dimensions the analogous construction requires a closed, but not exact, differential five-form. A simple example is the O(6)/O(5) generalization of the above (1+1)-dimensional construction.

> *Exercise:* Generalize the (1+1)-dimensional example to (3+1)-dimensions, considering the case of G=O(6), H=O(5), for which the field space is S^5. Derive the analog of Eq. (10), including the quantization condition on the overall normalization.

The existence of closed but not exact differential forms is formalized by the de-Rham cohomology space, $\mathrm{H}^{d+1}(G/H)$, whose dimensionality is given by the number of independent closed, but not exact, differential forms. For example, $\mathrm{H}^n(S^n) = \mathbb{R}$, where $\mathbb{R}$ denotes the real numbers (i.e., a space of dimension one), with $n = 3$ and $n = 5$ corresponding the S^3 and S^5 constructions in 1+1 and 3+1 spacetime dimensions.

Further generalizations are straightforward, but for Standard Model applicationss we focus attention on the example of $G = SU(n)_L \times SU(n)_R$ and $H = SU(n)_V$, with G/H = SU(n), $n \geq 3$. Here also $\mathrm{H}^5(\mathrm{SU}(N)) = \mathbb{R}$, and the closed but not exact five form defined for $U \in \mathrm{SU}(N)$ is

$$\Omega = \frac{-i}{240\pi^2} \mathrm{Tr}\left[(U^\dagger dU)^5\right] . \tag{11}$$

The normalization constant, $1/(240\pi^2)$, is the smallest value such that

$$\int_{M^5} \Omega = 2\pi \times \text{integer}\,, \tag{12}$$

for any closed five-dimensional manifold M^5. In particular, there exist closed manifolds for which this integer is unity. It is straightforward to see that Ω is invariant under G transformations, for which $U \to e^{i\epsilon_L} U e^{-i\epsilon_R}$. G is also readily seen to be closed,

$$d\Omega = \frac{-i}{240\pi^2} \mathrm{Tr}\left[dU^\dagger dU (U^\dagger dU)^4 - U^\dagger dU dU^\dagger dU (U^\dagger dU)^3 + \cdots + (U^\dagger dU)^4 dU^\dagger dU\right] = 0\,, \tag{13}$$

using the anticommutativity of differential one-forms. Finally, that Ω is not exact follows from the nonvanishing integral (12) taken over five-manifolds without boundary. Thus Ω satisfies our requirements, and we may define an action as

$$\Gamma = p \int_{M^5} \Omega\,, \tag{14}$$

where p is an integer, and where M^5 is chosen such that the image of spacetime in $SU(n)$ is the boundary of M^5. The normalization of Ω ensures that Γ is unique modulo 2π for any such choice of M^5.

> *Exercise:* Compute the explicit form of the interaction (14) through first nonzero order in $1/f_\pi$, using $U(x) = \exp[2i\pi^a(x)t^a/f_\pi]$. You should find
>
> $$\mathcal{L} = \frac{2p}{15\pi^2 f_\pi^5} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(t^a t^b t^c t^d t^e) \pi^a \partial_\mu \pi^b \partial_\nu \pi^c \partial_\rho \pi^d \partial_\sigma \pi^e\,.$$

Gauged WZW terms

Many applications involving the WZW term involve the inclusion of gauge fields. Two equivalent approaches to obtaining the gauged WZW term may be taken, with very different starting points. In the first approach, the starting point is the ungauged action, (14). By studying the variation of this action under local gauge transformations, we can introduce terms with gauge fields in a particular way such that the total variation is independent of pion fields. In the second approach, the starting point is a gauge anomaly, e.g. deduced from an underlying fermionic theory. From this perspective we can introduce pion fields and interactions such that the pionic action reproduces the gauge anomaly.

Brute force gauging

Let us recall the (1+1)-dimensional example with G/H=S^3. We may equivalently view this coset space as O(4)/O(3), or as U(2)/U(1). Taking the latter case, let us try to gauge all of the symmetries,

$$\Phi(x) \to e^{i(\epsilon+\epsilon_0)}\Phi\,, \tag{15}$$

where $\epsilon \equiv \epsilon^i\sigma^i$ with σ^i the Pauli matrices. Recall that our ungauged action is (subscript 0 denoting zero gauge fields)

$$\Gamma_0(\Phi) = \frac{p}{2\pi}\int_{M^3} \Phi^\dagger d\Phi d\Phi^\dagger d\Phi\,. \tag{16}$$

The variation of this action under the gauge transformation (15) is

$$\begin{aligned}\delta\Gamma_0 &= \frac{ip}{2\pi}\int_{M^3} d\epsilon^0 d\Phi^\dagger d\Phi + \Phi^\dagger d\epsilon\Phi d\Phi^\dagger d\Phi - \Phi^\dagger d\Phi\Phi^\dagger d\epsilon d\Phi + \Phi^\dagger d\Phi d\Phi^\dagger d\epsilon\Phi \\ &= \frac{ip}{4\pi}\int_{M^2} -2d\epsilon_0\Phi^\dagger d\Phi + \Phi^\dagger d\epsilon\Phi + d\Phi^\dagger d\epsilon\Phi\,.\end{aligned} \tag{17}$$

> *Exercise:* Derive the result (17). Notice the identities: $d(\Phi^\dagger\sigma^i\Phi)d\Phi^\dagger d\Phi = 0$, $d\Phi^\dagger\sigma^i d\Phi - d(\Phi^\dagger\sigma^i\Phi)\Phi^\dagger d\Phi + \Phi^\dagger\sigma^i\Phi d\Phi^\dagger d\Phi = 0$.

Introduce gauge fields A_0 and $A = A^i\sigma^i$, transforming as

$$\delta A_0 = d\epsilon_0\,, \quad \delta A = d\epsilon + i[\epsilon, A]\,, \tag{18}$$

and a contribution to the action whose variation thus cancels the $d\epsilon_0$ and $d\epsilon$ terms in Eq. (17),

$$\Gamma_1 = \frac{ip}{4\pi}\int_{M^2} 2A_0\Phi^\dagger d\Phi - \Phi^\dagger A d\Phi - d\Phi^\dagger A\Phi\,. \tag{19}$$

The residual variation of the combined action is

$$\delta(\Gamma_0+\Gamma_1) = \frac{p}{2\pi}\int_{M^2} \frac{1}{2}\mathrm{Tr}(Ad\epsilon) - A_0 d\epsilon_0 - A_0\Phi^\dagger d\epsilon\Phi - d\epsilon_0\Phi^\dagger A\Phi\,. \tag{20}$$

Finally, add a term whose variation cancels the Φ-dependent $d\epsilon_0$ and $d\epsilon$ terms in Eq. (20),

$$\Gamma_2 = \frac{p}{2\pi}\int_{M^2} A_0\Phi^\dagger A\Phi\,. \tag{21}$$

Defining the fully gauged action as

$$\Gamma_{\mathrm{WZW}} = \Gamma_0 + \Gamma_1 + \Gamma_2\,, \tag{22}$$

the variation is independent of Φ,

$$\delta\Gamma_{\mathrm{WZW}} = \frac{p}{4\pi}\int_{M^2} \mathrm{Tr}(\epsilon dA) - 2\epsilon_0 dA_0\,. \tag{23}$$

Anomaly integration

Suppose that we knew nothing about a theory except that when coupled to U(2) gauge fields, it produced the gauge anomaly in Eq. (23). Let us proceed to "integrate" this anomaly and compute a corresponding pion action.

In order to proceed, we must introduce the notion of counterterms, and a boundary condition for the anomaly integration. In particular, we require a form of the anomaly which vanishes for gauge transformations in the unbroken subgroup H, in the presence of arbitrary gauge fields belonging to the full group G. The corresponding Bardeen counterterm for the present case is

$$\Gamma_c(A, A_0) = \frac{-p}{2\pi}\int_{M^2} A_0\langle A\rangle = -\Gamma_{WZW}(A, A_0, \Phi = \langle\Phi\rangle) \tag{24}$$

where $(\epsilon + \epsilon)\langle\Phi\rangle = 0$ defines the unbroken subgroup. Since the result of integration will be a series of terms involving pions, for consistency, this counterterm must be precisely the gauged WZW action obtained previously, restricted to vanishing pion fields; this is indeed seen to be the case.

The anomaly with counterterm now takes the form,

$$\delta(\Gamma_{WZW} + \Gamma_c) = \frac{p}{4\pi}\int_{M^2} \mathrm{Tr}\Big[\langle\epsilon dA\rangle - 2\langle\epsilon\rangle dA_0 + 2i\langle\epsilon A - A\epsilon\rangle A_0 - \epsilon_0 dA_0 + 2\epsilon_0\langle dA\rangle\Big] \equiv \int_{M^2}\epsilon^a \mathcal{A}^a\,. \tag{25}$$

Exercise: Verify that the anomaly (25) is consisent and vanishes on the unbroken subgroup.

Let us introduce pion fields defined to transform as a nonlinear realization,

$$e^{i\pi} \to e^{i(\epsilon+\epsilon_0)} e^{i\pi} e^{-i\epsilon'(\epsilon+\epsilon_0,\pi)}\,. \tag{26}$$

Define also the quantity A^t as a field-dependent gauge transformation of A,

$$A^t = e^{-it\pi}(A + id)e^{it\pi}\,. \tag{27}$$

The integration of the anomaly is then given by the compact expression,

$$\Gamma_{WZW} + \Gamma_c = \int_{M^2}\int_0^1 dt\, \pi^a \mathcal{A}^a[A^t]\,. \tag{28}$$

Consider again the special case of S^3 in 1+1 dimensions. The pion fields may be equivalently represented as

$$\Phi(x) = e^{i\pi(x)}\langle\Phi\rangle\,. \tag{29}$$

Let us choose the particular basis

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \epsilon + \epsilon_0 = \begin{pmatrix} \epsilon_B & \epsilon_{C^+} \\ \epsilon_{C^-} & \epsilon_D \end{pmatrix}, \tag{30}$$

and in this basis write

$$\pi = \begin{pmatrix} 0 & h^+ \\ h^- & 2\eta \end{pmatrix}, \quad A + A_0 = \begin{pmatrix} B & C^+ \\ C^- & D \end{pmatrix}. \tag{31}$$

The explicit anomaly variation is

$$\delta\Gamma_{WZW} = \frac{p}{4\pi}\int_{M^2} d\epsilon_B D + d\epsilon_D B - d\epsilon_{C^+}C^- \tag{32}$$

and with the addition of the counterterm,

$$\Gamma_c = -\frac{p}{4\pi}\int_{M^2} BD\,, \tag{33}$$

the variation becomes,

$$\delta(\Gamma_{WZW} + \Gamma_c) = \frac{p}{4\pi}\int_{M^2} -2\epsilon_D dB + \epsilon_{C^+}dC^- + \epsilon_{C^-}dC^+ - i(\epsilon_{C^+}C^- - \epsilon_{C^-}C^+)(B + D)\,. \tag{34}$$

In particular, the anomaly vanishes for $\epsilon_B = 0$ regardless of whether $C^\pm$ and D fields are present. The explicit expansion of A^t yields

$$A^t = \begin{pmatrix} B & C^+ \\ C^- & D \end{pmatrix} + it\begin{pmatrix} h^-C^+ - h^+C^- & (B-D)h^+ + 2\eta C^+ + idh^+ \\ (D-B)h^- - 2\eta C^- + idh^- & h^+C^- - h^-C^+ + 2id\eta \end{pmatrix} + \mathcal{O}(t^2)\,. \tag{35}$$

Finally, substitution in the integration formula yields,

$$\Gamma_{WZW} + \Gamma_c = \frac{p}{\pi}\int_{M^2} -Bd\eta + \frac{1}{4}(C^+dh^- + C^-dh^+) - \frac{i}{4}(B + D)(C^+h^- - C^-h^+) + \mathcal{O}(\pi^2) \tag{36}$$

Exercise: Express the result of brute-force gauging (22) in the basis (31), verifying that these actions are identical through first order in pions.