Nonrelativistic limit

We have considered full-theory matching to an effective theory so far in cases where the difference between full and effective theories is a momentum shell ($\mathcal{L}_\lambda \to \mathcal{L}_{b\Lambda}$) or an entire particle state (e.g., integrating out massive vector bosons in Fermi electroweak theory). Another important example is the case where antiparticle components of a fermion are not relevant, e.g., electrons in atoms, or heavy quarks in hadrons.

We derived the effective lagrangian (cf. lecture 7 and associated exercise) for a heavy Dirac fermion. In covariant notation, at tree level,

$$\mathcal{L} = \bar{h}_v \left[iv\cdot D + \frac{(i\not{D}_\perp)^2}{2M} + \dots \right] h_v \,, \tag{1}$$

where $D_\perp^\mu = D^\mu - v\cdot D v^\mu$. Let us specialize to $v^\mu = (1,0,0,0)$, and write

$$\mathcal{L} = \psi^\dagger \left[iD_t + \frac{\boldsymbol{D}^2}{2M} + \dots \right]\psi = \psi^\dagger \left[i\partial_t + gA^0 + \frac{\boldsymbol{\partial}^2}{2M} + \frac{ig}{2M}\left(\boldsymbol{\partial}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\partial}\right) + \frac{g}{2M}\boldsymbol{\sigma}\cdot\boldsymbol{B} + \dots \right]\psi \,, \tag{2}$$

where ψ is a two-component spinor field. Consider the conjugate momentum field,

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = i\psi^\dagger \,, \tag{3}$$

and the Hamiltonian (density),

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger \left(-gA^0 - \frac{\boldsymbol{\partial}^2}{2M} + \dots \right)\psi \,. \tag{4}$$

To quantize the free theory ($g=0$), let us write

$$\psi(\boldsymbol{x},t) = \int \frac{d^3p}{(2\pi)^3} \sum_s a_{\boldsymbol{p},s} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}} \chi_s \,, \quad \{a_{\boldsymbol{p},s}, a^\dagger_{\boldsymbol{p}',s'}\} = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ss'} \,, \tag{5}$$

where $E_{\boldsymbol{p}} = \boldsymbol{p}^2/2M$ and χ_s runs over a basis of two-component spinors. Let us view A^μ as a static background field, e.g., a Coulomb field seen by an atomic electron.

Exercise: Show that ψ and π obey canonical anticommutation relations, $\{\psi(\boldsymbol{x},t), \pi(\boldsymbol{x}',t)\} = i\delta^3(\boldsymbol{x}-\boldsymbol{x}')$.

Expanding the Hamiltonian, we find

$$\begin{aligned} H = \int d^3x \mathcal{H}(\boldsymbol{x},0) &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{\boldsymbol{p}^2}{2M} a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} - g\sum_s \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} \int d^3x a^\dagger_{\boldsymbol{p}',s} a_{\boldsymbol{p},s} e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} A^0(\boldsymbol{x}) + \dots \,, \\ &= \sum_{s,s'} \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} a^\dagger_{\boldsymbol{p}',s'} a_{\boldsymbol{p},s} \langle \boldsymbol{p}', s'| \left(\frac{\boldsymbol{p}^2}{2M} - gA^0(\boldsymbol{x}) + \dots \right) |\boldsymbol{p}, s\rangle \,. \end{aligned} \tag{6}$$

Here

$$|\boldsymbol{p}, s\rangle \equiv a^\dagger_{\boldsymbol{p},s} |\mathrm{vac}\rangle \,. \tag{7}$$

When only states of definite particle number are relevant, e.g. states containing exactly one electron for applications to atoms, the additional structure involve creation and annihilation operators is redundant, and we may simply identify

$$H = \frac{\boldsymbol{p}^2}{2M} - gA^0(\boldsymbol{x}) + \dots \,, \tag{8}$$

as the Hamiltonian acting on quantum mechanical Hilbert space in the standard way.

Exercise: Check that (6) and (8) have identical matrix elements between one-particle states.