Renormalization: Fermi theory of weak interactions

Let us use the formalism we have constructed to analyze a practical problem in the standard model, namely the renormalization of effective operators mediating weak interactions.

1 Matching

As in our toy QED-scalar example, the phenomena of operator mixing occurs, and logarithms in ratios of scales are summed by RG equations. We will take for granted the Feynman rules for the Standard Model, which yield for the tree-level amplitude mediating b quark decay to a final state of c and d quarks and u antiquark,

$$\begin{aligned} i\mathcal{M} &= \bar{u}^{(c)} \frac{ig_2}{2\sqrt{2}} \gamma^\mu (1-\gamma_5) V_{cb}^* u^{(b)} \frac{-i}{q^2 - m_W^2} \bar{u}^{(d)} \frac{ig_2}{2\sqrt{2}} \gamma_\mu (1-\gamma_5) V_{ud} v^{(u)} \,. \\ &= -ic_1 \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} \bar{u}^{(c)} \gamma^\mu (1-\gamma_5) u^{(b)} \bar{u}^{(d)} \gamma_\mu (1-\gamma_5) v^{(u)} + \dots \,, \end{aligned} \tag{1}$$

where

$$c_1 = 1 \tag{2}$$

We obtain the same amplitude from the effective lagrangian,

$$\mathcal{L}_{\rm eff} = -\frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [c_1 O_1 + c_2 O_2] \,, \tag{3}$$

where

$$O_1 = \bar{c}_i \gamma^\mu (1-\gamma_5) b_i \bar{d}_j \gamma_\mu (1-\gamma_5) u_j \,, \quad O_2 = \bar{c}_i \gamma^\mu (1-\gamma_5) b_j \bar{d}_j \gamma_\mu (1-\gamma_5) u_i \,, \tag{4}$$

and

$$c_1 = 1 \,, \quad c_2 = 0 \,. \tag{5}$$

The i, j subscripts are color indices. We have assumed Fierz rearrangement to write operators in this form. *Exercise:* Give a symmetry reason why the basis O_1, O_2 is complete under QCD renormalization.

We may perform the matching to higher order in QCD perturbation theory, obtaining c_i as a perturbative expansion in g_s, the strong coupling constant.

2 Anomalous dimension

Let us use dimensional regularization in the $\overline{\rm MS}$ scheme. Proceeding as in the toy example, we may compute the one-loop diagrams contributing UV divergences to the matrix elements of O_i, from which we deduce the operator renormalization constants, and hence the anomalous dimensions.

We find for the three basic diagrams with insertion of O_1,

$$\begin{aligned} \text{diag.}\,1 &= g^2 I(\epsilon,\lambda) t^a t^a \gamma^\mu (1-\gamma_5) \otimes \gamma_\mu (1-\gamma_5) \,, \\ \text{diag.}\,2 &= g^2 I(\epsilon,\lambda) \left(-\frac{1}{4}\right) t^a \gamma^\mu (1-\gamma_5) \gamma^\beta \gamma^\alpha \otimes t^a \gamma_\mu (1-\gamma_5) \gamma_\beta \gamma_\alpha \,, \\ \text{diag.}\,3 &= g^2 I(\epsilon,\lambda) \frac{1}{4} t^a \gamma^\mu (1-\gamma_5) \gamma^\beta \gamma^\alpha \otimes t^a \gamma_\alpha \gamma_\beta \gamma_\mu (1-\gamma_5) \,, \end{aligned} \tag{6}$$

Using standard identities for Dirac matrices, and the $SU(N)$ matrix identities,

$$(t^a)_{ij} (t^a)_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \,, \quad t^a t^a = C_F \mathbb{1} \,, \quad C_F = (N^2-1)/2N \,, \tag{7}$$

and including a factor of two from reflected diagrams, we have

$$\langle O_1 \rangle = 2g^2 I(\epsilon,\lambda) \left[\frac{11}{6} \langle O_1 \rangle^{\rm tree} - \frac{3}{2} \langle O_2 \rangle^{\rm tree} \right] . \tag{8}$$

The integral I we have encountered before,

$$I(\epsilon,\lambda) = -i \int \frac{d^d L}{(2\pi)^d} \frac{1}{(L^2+i0)(L^2-\lambda^2+i0)} = \frac{1}{(4\pi)^2} [(4\pi)^\epsilon \Gamma(1+\epsilon)] \lambda^{-2\epsilon} \frac{1}{\epsilon(1-\epsilon)} = \frac{1}{(4\pi)^2} \frac{1}{\epsilon} + \dots . \tag{9}$$

Repeating the exercise for O_2,

$$\langle O_2 \rangle = 2g^2 I(\epsilon,\lambda) \left[\frac{11}{6} \langle O_2 \rangle^{\rm tree} - \frac{3}{2} \langle O_1 \rangle^{\rm tree} \right] . \tag{10}$$

Exercise: Derive (8) and (10).

The wavefunction renormalization factor for quarks in QCD is similar to that for electrons in QED, but with an extra color factor,

$$\begin{aligned} Z_q^{\rm onshell} &= 1 - g^2 C_F I(\epsilon,\lambda) , \\ Z_q^{\rm MS} &= 1 - \frac{g^2}{(4\pi)^2} \frac{4}{3\epsilon} . \end{aligned} \tag{11}$$

The operator renormalization constants are defined to absorb possible divergences in addition to usual field and coupling renormalization (this is the dim.reg., MS, implementation of the momentum shell integration that we studied previously)

$$\begin{aligned} (Z_q^{\frac12})^4 \langle O_1 \rangle &= \langle O_1 \rangle^{\rm tree} \left[1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \right] + \langle O_2 \rangle^{\rm tree} \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (-3) , \\ (Z_q^{\frac12})^4 \langle O_2 \rangle &= \langle O_1 \rangle^{\rm tree} \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (-3) + \langle O_2 \rangle^{\rm tree} \left[1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \right] , \end{aligned} \tag{12}$$

from which we read off,

$$\hat{Z}_{\mathcal{O}} = \mathbb{1} + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left(\begin{array}{cc} 1 & -3 \\ -3 & 1 \end{array} \right) , \tag{13}$$

and hence

$$\hat\gamma = -g \frac{\partial}{\partial g} \hat{Z}_{\mathcal{O},1} = \frac{\alpha_s}{4\pi} \hat\gamma_0 + \left(\frac{\alpha_s}{4\pi} \right)^2 \hat\gamma_1 + \dots = \frac{\alpha_s}{2\pi} \left(\begin{array}{cc} -1 & 3 \\ 3 & -1 \end{array} \right) . \tag{14}$$

3 Renormalization

Having computed the anomalous dimension of the operators, we may proceed to solve for the coefficients renormalized at low scales, say of order the heavy quark masses. The coefficients obey the renormalization equation,

$$\frac{d}{d\log\mu} \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right) = \hat\gamma^T \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right) , \tag{15}$$

or, diagonalizing,

$$\frac{d}{d\log\mu} \left(\begin{array}{c} c_1 + c_2 \\ c_1 - c_2 \end{array} \right) = \frac{\alpha_s}{4\pi} \left(\begin{array}{cc} 4 & 0 \\ 0 & -8 \end{array} \right) \left(\begin{array}{c} c_1 + c_2 \\ c_1 - c_2 \end{array} \right) , \tag{16}$$

The strong coupling obeys the equation (we shall take this as input; it is derived from the gluon field strength renormalization as in the QED case, but accounting for the nonabelian nature of the gluon)

$$\frac{d\alpha_s}{d\log\mu} = -\frac{\beta_0}{2\pi} \alpha_s^2 + \dots . \tag{17}$$

We thus have for each coefficient eigenvalue,

$$\frac{dc_\pm}{c_\pm} = \gamma_\pm d\log\mu = \left(\frac{\alpha_s}{4\pi} \gamma_{\pm,0} + \dots \right) \frac{d\alpha_s}{-\frac{\beta_0}{2\pi}\alpha_s^2 + \dots} = -\frac{\gamma_{\pm,0}}{2\beta_0} \frac{d\alpha_s}{\alpha_s} + \dots , \tag{18}$$

where $\gamma_{\pm,0} = 2(-1 \pm 3)$ The leading order solution to the renormalization group evolution is thus

$$c_\pm(\mu) = c_\pm(m_W) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_W)} \right]^{-\frac{\gamma_{\pm,0}}{2\beta_0}} , \tag{19}$$

Exercise: Solve (17) to find

$$\alpha_s(\mu) = \frac{\alpha_s(m_W)}{1 + \alpha_s(m_W) \frac{\beta_0}{2\pi} \log \frac{\mu}{m_W}} . \tag{20}$$

Given the leading order matching, $c_1 = 1$, $c_2 = 0$ at $\mu = m_W$, and $\alpha_s(m_W) \approx 0.12$, find $\alpha_s(m_b)$ and the solution for $c_i(m_b)$, using $m_b \approx 4.5\,\mathrm{GeV}$).