Renormalization: operator renormalization and log summation

In practical applications, we often wish to perform perturbation theory for a process occuring at a low scale (μ), using a theory defined at a high scale ($\Lambda \gg \mu$). Provided that couplings are vanishingly small, we may proceed with the usual perturbation theory program. However, in practice, the large ratio of scales can invalidate a perturbative treatment. For example, in our analysis of $\lambda\phi^4$ theory, the product $\lambda(\Lambda)\log(\Lambda/\mu)$ may become of order unity, even if $\lambda(\Lambda)$ is small. To obtain sensible predictions we must "resum" such logarithms. A similar situation arises when applying perturbative QCD to processes at a scale that is low compared to high-energy collisions or to the electroweak scale; here, terms $\alpha_s(\mu)\log(\Lambda/\mu)$ must again be summed to obtain sensible predictions. The renormalization program is usually phrased in terms of effective field theory, typically involving three steps: 1) constructing an effective theory by integrating out degrees of freedom (high energy modes, massive fields, or antiparticle components of heavy fermions); 2) matching full theory to effective theory; and 3) renormalization of couplings in the effective theory to the low scale of interest.

1 Effective operators

As a simple example of effective operators, consider two massless fermions interacting with a scalar field and abelian gauge field,

$$\mathcal{L} = \sum_{i=1}^{2} \bar\psi_i (i\partial\!\!\!/ + gA\!\!\!/)\psi_i - \frac14 (F_{\mu\nu})^2 + \frac12 (\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 + \kappa\phi\sum_{i=1}^{2}\bar\psi_i\psi_i \,. \tag{1}$$

(This is a toy example for low-energy Fermi theory of weak interactions, with ϕ playing the role of massive $W^\pm$ and Z^0 bosons, the photon playing the role of gluons and ψ_i the role of quarks and leptons.) At energies small compared to M we may integrate out ϕ. The effective Lagrangian induced by ϕ exchange is

$$\mathcal{L}_{\rm eff} = \bar\psi(i\partial\!\!\!/ + gA\!\!\!/)\psi - \frac14 (F_{\mu\nu})^2 + c_1 \bar\psi_1\psi_1\bar\psi_2\psi_2 + \dots\,, \tag{2}$$

where we focus on interactions mediating ψ_1-ψ_2 scattering (there are also ψ_1-ψ_1 and ψ_2-ψ_2 interactions) and the tree level matching condition betweeen theories (1) and (2) is

$$c_1 = \frac{\kappa^2(\Lambda)}{M^2} + \dots\,. \tag{3}$$

We may determine the coefficients of operators in the effective lagrangian by matching full and effective theories for arbitrary low-energy external states.

Considering (1) defined with regulator Λ, all couplings refer to this renormalization scale. To account for possible large logarithms in perturbation theory, we should analyze the behavior of (2) under renormalization. Thus consider as before, the process of integrating over a momentum shell. The four point function in the full theory (including modes up to Λ) is

$$(Z_{\rm full}^{\frac12})^4 i\mathcal{M}_{\rm full} = ic(\Lambda)(Z_{\rm full}^{\frac12})^4\left\{1\otimes 1 + 2g^2(\Lambda)I(\Lambda,\lambda)\left[d1\otimes 1 + \frac{2}{d}\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\right]\right\} \tag{4}$$

In the effective theory, obtained by integrating out modes between $b\Lambda$ and Λ,

$$(Z_{\rm eff})^{\frac12})^4 i\mathcal{M}_{\rm eff} = ic(b\Lambda)(Z_{\rm eff})^{\frac12})^4\left\{1\otimes 1 + 2g^2(b\Lambda)I(b\Lambda,\lambda)\left[d1\otimes 1 + \frac{2}{d}\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\right]\right\} \tag{5}$$

The Z factor is familiar from QED,

$$1 - Z^{-1} = \frac{d}{dp\!\!\!/}\Sigma(p)\Big|_{p=0} = \frac{d}{dp\!\!\!/}\Big|_{p=0}\left[-ig^2\int\frac{d^dL}{(2\pi)^d}\frac{\gamma^\mu(L\!\!\!/ + p\!\!\!/)\gamma_\mu}{(L+p)^2(L^2-\lambda^2)}\right] = -g^2\frac{(d-2)^2}{d}I(\Lambda,\lambda) \tag{6}$$

So far expressions refer to arbitrary spacetime dimension d, for later, more elegant, treatment in dimensional regularization. Specializing to $d=4$, the basic integral we require is

$$I(\Lambda,\lambda) \equiv -i\int \frac{d^4L}{(2\pi)^4}\bigg|_{|L_E|<\Lambda} \frac{1}{L^2}\frac{1}{L^2-\lambda^2} = \frac{2}{(4\pi)^2}\int_0^\Lambda dL\,\frac{L}{L^2+\lambda^2} = \frac{1}{16\pi^2}\log\frac{\Lambda^2+\lambda^2}{\lambda^2} = \frac{1}{16\pi^2}\left(\log\frac{\Lambda^2}{\lambda^2}+\mathcal{O}(\lambda^2)\right)\,, \tag{7}$$

where we have regulated with a photon mass λ. Finally, equating full and effective theories we find the conditions,

$$\begin{aligned} c_1(b\Lambda) &= c_1(\Lambda)\left[1+6\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\right]+\dots\,,\\ c_2(b\Lambda) &= c_1(\Lambda)\left[\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\right]+\dots\,. \end{aligned} \tag{8}$$

In particular, we require the inclusion of a new operator in (2) in order to perform the matching,

$$\begin{aligned} O_1 &= \bar\psi_1\psi_1\bar\psi_2\psi_2\,,\\ O_2 &= \bar\psi_1\sigma^{\mu\nu}\psi_1\bar\psi_2\sigma_{\mu\nu}\psi_2\,. \end{aligned} \tag{9}$$

Operator O_2 has been “induced” by renormalization.

2 Anomalous dimension

Having induced the opeator O_2, in succeeding renormalization steps we must also consider its effects. In fact, we find that yet one further operator is induced,

$$O_3 = \bar\psi_1\gamma_5\psi_1\bar\psi_2\gamma_5\psi_2\,. \tag{10}$$

Exercise: Give a symmetry reason why the basis O_1, O_2, O_3 is complete.

Repeating the momentum-shell integration step, we find the amplitude for arbitrary c_i,

$$\begin{aligned} (Z_\psi^{\frac12})^4\mathcal{M} = \left[1-g^2I(\Lambda,\lambda)\right]^2 \Big\{ & c_1 1\otimes 1 + c_2\sigma^{\mu\nu}\otimes\sigma_{\mu\nu} + c_3\gamma_5\otimes\gamma_5 \\ & + g^2 I(\Lambda,\lambda)\Big[c_1\big(8\,1\otimes 1+\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\big) + c_2\big(24\,1\otimes 1 + 24\,\gamma_5\otimes\gamma_5\big) + c_3\big(8\,\gamma_5\otimes\gamma_5+\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\big)\Big] \end{aligned} \tag{11}$$

Equating this expression for theories with cutoff Λ and $b\Lambda$, we find the generalization of (8)

$$\begin{aligned} c_1(b\Lambda) &= c_1(\Lambda)\left[1+6\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\right] + 24c_2(\Lambda)\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\,,\\ c_2(b\Lambda) &= c_1(\Lambda)\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2} + c_2(\Lambda)\left[1-2\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\right] + c_3(\Lambda)\frac{g^2}{16\pi^2}\log\frac{1}{b^2}\,,\\ c_3(b\Lambda) &= 24c_2(\Lambda)\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2} + c_3(\Lambda)\left[1+6\frac{g^2(\Lambda)}{16\pi^2}\log\frac{1}{b^2}\right] \end{aligned} \tag{12}$$

Identifying $\mu=b\Lambda$, we find the differential equations,

$$\frac{d}{d\log\mu}\vec{c}(\mu) = \hat\gamma^T\vec{c}(\mu)\,, \tag{13}$$

where $\hat\gamma$ is the “anomalous dimension” matrix (the transpose is conventional: $\hat\gamma$ with a minus sign appears as the anomalous dimension of *operators* versus coefficients), which to leading order in g^2 is

$$\hat\gamma^T = -\frac{\alpha}{2\pi}\begin{pmatrix} 6 & 24 & 0\\ 1 & -2 & 1\\ 0 & 24 & 6\end{pmatrix}\,. \tag{14}$$

In terms of the combinations

$$c'_1 = c_1 - c_3\,, \quad c'_2 = c_1 + c_3 - 12c_2\,, \quad c'_3 = c_1 + c_3 + 4c_2\,, \tag{15}$$

the anomalous dimension matrix is diagonal,

$$\hat{\gamma}' = -\frac{\alpha}{2\pi}\mathrm{diag}(6, -6, 10)\,. \tag{16}$$

For each eigenvalue, we write the anomalous dimension as an expansion in α,

$$\gamma' = \gamma_0 \frac{\alpha}{4\pi} + \dots\,, \tag{17}$$

Recall the running of the electromagnetic coupling,

$$\frac{d\alpha}{d\log\mu} = \frac{\beta_0}{2\pi}\alpha^2 + \dots, \tag{18}$$

so that the coefficient eigenvalues obey

$$\frac{dc}{c} = \gamma d\log\mu = \gamma\frac{d\alpha}{\beta} = \frac{\gamma_0}{2\beta_0}\frac{d\alpha}{\alpha} + \dots\,, \implies c(\mu) = c(\Lambda)\left[\frac{\alpha(\mu)}{\alpha(\Lambda)}\right]^{\frac{\gamma}{2\beta_0}} + \dots\,. \tag{19}$$

When the right hand side of this equation is expanded in either $\alpha(\mu)$ or $\alpha(\Lambda)$, powers of $\log(\Lambda/\mu)$ appear. In applications involving the QCD analog of this example, such logarithms can overcome the suppression by α_s, the strong coupling analog of α, and must be summed in this way to obtain meaningful results.

2.1 Computation in dimensional regularization

Recall from our study of QED renormalization in dimensional regularization in minimal subtraction, that we may compute beta functions and related quantities by isolating $1/\epsilon$ UV divergences. For the β function, we have $g_{\rm bare} = Z_g \mu^\epsilon g(\mu)$, where $Z_g = Z_A^{-\frac{1}{2}}$, with Z_A the wavefunction renormalization factor of the photon field, as determined by the 1PI photon two-point function,

$$\begin{aligned} Z_A^{\rm onshell} &= 1 + \frac{g_{\rm bare}^2}{(4\pi)^2}[(4\pi)^\epsilon\Gamma(1+\epsilon)]m_{\rm bare}^{-2\epsilon}\left(-\frac{4}{3\epsilon} + \frac{2}{3}\right)\,, \\ Z_A^{\rm MS} &= 1 + \frac{g^2}{(4\pi)^2}\left(-\frac{4}{3\epsilon}\right)\,. \end{aligned} \tag{20}$$

Let us write,

$$Z_g = 1 + \frac{1}{\epsilon}Z_{g,1} + \frac{1}{\epsilon^2}Z_{g,2} + \dots\,. \tag{21}$$

Then because bare quantities are independent of μ,

$$\frac{d}{d\log\mu}g_{\rm bare} = 0 \implies \beta(g,\epsilon) = -\epsilon g(1 + Z_g^{-1} g \frac{\partial}{\partial g}Z_g)^{-1}\,, \tag{22}$$

where

$$\beta(g,\epsilon) \equiv \frac{d}{d\log\mu}g\,. \tag{23}$$

Since $Z_g - 1$ contains only powers of $1/\epsilon$, the finiteness of $\lim_{\epsilon\to 0}\beta(g,\epsilon)$ implies

$$\beta(g) \equiv \lim_{\epsilon\to 0}\beta(g,\epsilon) = g^2\frac{\partial}{\partial g}Z_{g,1} = -\frac{1}{2}g^2\frac{\partial}{\partial g}Z_{A,1} = \frac{g^3}{(4\pi)^2}\left(\frac{4}{3}\right) + \dots\,, \tag{24}$$

where we have used (20). We thus find,

$$\frac{d\alpha}{d\log\mu} = \frac{g}{2\pi}\beta(g) = \frac{2\alpha^2}{3\pi} \equiv -\frac{\beta_0}{2\pi}\alpha^2\,, \tag{25}$$

where $\beta_0(\mathrm{QED}) = -4/3$.

Similarly for the μ dependence of renormalized operators, let us write

$$\mathcal{O}_{\rm bare,\, i} = Z_{ij}\mathcal{O}(\mu)_j \, , \tag{26}$$

so that

$$\frac{d}{d\log\mu}\mathcal{O}_i = -\gamma_{ij}\mathcal{O}_j \, , \quad \hat{\gamma} \equiv \hat{Z}_{\mathcal{O}}^{-1}\frac{d}{\log\mu}\hat{Z}_{\mathcal{O}} \, , \tag{27}$$

Exercise: Derive the expression for γ in terms of Z.

In dimensional regularization with minimal subtraction, we have

$$Z_{\mathcal{O}} = 1 + \frac{1}{\epsilon}Z_{\mathcal{O},1} + \frac{1}{\epsilon^2}Z_{\mathcal{O},2} + \dots \, , \tag{28}$$

and hence by an argument similar to that for the beta function, we have to all orders in perturbation theory,

$$\hat{\gamma} = -g\frac{\partial}{\partial g}Z_{\mathcal{O},1} \, . \tag{29}$$

Exercise: Derive (29), using (22), (28) and (29).