Renormalization: integrating out offshell modes

1 Integrating over a momentum shell

Consider a scalar (for simplicity) field theory regulated by a cutoff $|k_E| \le \Lambda$ in Euclidean momentum space. Taking $t^0 = -it^0_E$ we have

$$iS \to -S_E^{(\Lambda)}(\phi) = -\int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \right], \tag{1}$$

with

$$\phi(x) = \int \frac{d^dp}{(2\pi)^d}_{|p|<\Lambda} \tilde{\phi}(p). \tag{2}$$

For physical processes with energy $E \ll \Lambda$, look to obtain the same physical predictions from a theory not containing momentum modes with $p \sim \Lambda$.

Let us phrase this as a matching computation, considering (1) as the "full" theory, and for the effective theory (having rescaled to canonically normalized kinetic term):

$$S_E^{(b\Lambda)}(\phi) = \int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m'^2\phi^2 + \frac{\lambda'}{4}\phi^4 + \left[\phi^6,\, \phi^2(\partial_\mu\phi)^2,\, \phi^2\partial^2\phi^2,\, \phi\partial^4\phi\right] + \dots \right]. \tag{3}$$

Let us work in perturbation theory assuming all couplings (including mass) remain small. The amputated two-point function in the full theory is

$$-\Sigma(k)_{\rm full} = -m^2 - \frac{\lambda}{2}\int \frac{d^dL}{(2\pi)^d}_{|L|\le\Lambda} \frac{1}{L^2} + \dots = -m^2 - \frac{\lambda}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\Lambda dL\, L^{d-3} + \dots, \tag{4}$$

while in the effective theory,

$$-\Sigma(k)_{\rm eff} = -m'^2 - \frac{\lambda'}{2}\int \frac{d^dL}{(2\pi)^d}_{|L|\le b\Lambda} \frac{1}{L^2} + \dots = -m'^2 - \frac{\lambda'}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^{b\Lambda} dL\, L^{d-3} + \dots, \tag{5}$$

The physical mass is given by

$$m_{\rm phys} = \Sigma(m_{\rm phys})_{\rm full} = \Sigma(m_{\rm phys})_{\rm eff}, \tag{6}$$

so that

$$\begin{aligned} m'^2 &= m^2 + \frac{\lambda}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_{b\Lambda}^\Lambda dL\, L^{d-3} - \frac{\lambda'-\lambda}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^{b\Lambda} dL\, L^{d-3} + \dots \\ &= m^2 + \frac{\lambda\Lambda^{d-2}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \frac{1}{d-2}(1-b^{d-2}) - \frac{\lambda'-\lambda}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \frac{b^{d-2}\Lambda^{d-2}}{d-2} . + \dots . \end{aligned} \tag{7}$$

Note that since $\Sigma(k^2)$ is independent of k^2 to the order we are working (the coupling constants of higher dimension operators in (3) are induced only at higher order), the onshell Z factors are trivial ($Z = 1$).

The zero-momentum limit of the amputated four-point function involves the tree level contribution and three permutations of the basic one-loop diagram

$$-\mathcal{M}_{4,\rm full} = -\lambda + 3\cdot\frac{1}{2}\lambda^2 \int \frac{d^dL}{(2\pi)^d}_{|L|\le\Lambda} \frac{1}{L^2}\frac{1}{L^2} + \dots . \tag{8}$$

In the effective theory,

$$-\mathcal{M}_{4,\rm eff} = -\lambda' + 3\cdot\frac{1}{2}\lambda'^2 \int \frac{d^dL}{(2\pi)^d}_{|L|\le\Lambda} \frac{1}{L^2}\frac{1}{L^2} + \dots . \tag{9}$$

Thus, since the Z factors are trivial to this order,

$$\begin{aligned}\lambda' &= \lambda - 3\frac{\lambda^2}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_{b\Lambda}^{\Lambda} dL\, L^{d-5} + \dots \\ &= \lambda - 3\frac{\lambda^2\Lambda^{d-4}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\frac{1}{d-4}(1-b^{d-4}) + \dots .\end{aligned} \tag{10}$$

1.1 Rescaling

Relations (7) and (10) relate the couplings in the theory (1) defined with $|p| \le \Lambda$ to the couplings in the theory (3) defined with $|p| \le b\Lambda \equiv \mu$. We may rescale coordinates to consider the latter theory on the same space of variables as the original theory,

$$k = bk', \quad x = b^{-1}x', \quad \phi(x) = \phi'(x'). \tag{11}$$

Thus

$$S_E^{(\mu)}(\phi) = b^{-d}\int d^dx'\left[\frac{1}{2}b^2(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{\lambda'}{4}\phi'^4 + \left[\phi'^6,\, b^2\phi'^2(\partial'_\mu\phi')^2,\, b^2\phi'^2\partial'^2\phi'^2,\, b^4\phi'\partial'^4\phi'\right] + \dots\right]. \tag{12}$$

Rescaling $\phi' = b^{d/2-1}\phi''$, then dropping primes we have finally

$$S_E^{(\mu)}(\phi) = \int d^dx'\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\bar{m}^2\phi^2 + \frac{\bar{\lambda}}{4}\phi^4 + \dots\right]. \tag{13}$$

Specializing to four dimensions,

$$\begin{aligned}\bar{m}^2(\mu) &= \frac{\Lambda^2}{\mu^2}\left[\bar{m}^2(\Lambda) + \frac{\bar{\lambda}(\Lambda)}{32\pi^2}(\Lambda^2-\mu^2) + \dots\right], \\ \bar{\lambda}(\mu) &= \bar{\lambda}(\Lambda) - \frac{3\bar{\lambda}^2(\Lambda)}{16\pi^3 2}\log\frac{\Lambda}{\mu} + \dots ,\end{aligned} \tag{14}$$

where $\bar{\lambda}(\Lambda) = \lambda$ and $\bar{m}^2(\Lambda) = m^2$.

2 Renormalization flows

2.1 Scalar mass divergence

From (4), the physical mass is given by the sum of a bar mass term and a (in four dimensions) quadratically divergent (at $\Lambda \to \infty$) perturbation. It appears that a fine tuning is necessary in order to obtain a physical mass $\ll \Lambda$. In the Standard Model, the Higgs field mass parameter similarly receives divergent corrections (in the limit where the regulator scale is taken large) from loops involving Standard Model fermions, gauge fields and Higgs scalars. Explaining why $m_H \ll \Lambda_{\rm NP}$, where $\Lambda_{\rm NP}$ is the scale to which new physics has been excluded, is called the "hierarchy" problem.

2.2 Landau pole

Consider the leading order renormalization equation,

$$\frac{d}{d\log\mu}\lambda(\mu) = \frac{3\lambda^2(\mu)}{16\pi^2}, \tag{15}$$

with solution

$$\frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2}d\log\mu \implies \lambda(\mu) = \frac{\lambda(\Lambda)}{1 + \frac{3}{16\pi^2}\lambda(\Lambda)\log\frac{\Lambda}{\mu}}. \tag{16}$$

Conversely,

$$\lambda(\Lambda) = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\log\frac{\Lambda}{\mu}}. \tag{17}$$

Thus for a fixed μ, the coupling $\lambda(\Lambda)$ that is required to induce a given $\lambda(\mu)$ diverges. This formal singularity where perturbation theory breaks down is called a "Landau pole". Note that even in the absence of explicit mass scales defining the low-energy theory, the Landau pole phenomenon defines a high energy scale.

2.3 Fixed point

Consider the case $d \neq 4$. The renormalization equation for the coupling (consider the dimensionless combination $\lambda(\mu) = \Lambda^{(d-4)/2}\bar{\lambda}(\mu)$) becomes

$$\beta^{(\lambda)} \equiv \frac{d}{d\log\mu}\lambda(\mu) = (d-4)\lambda(\mu) + \frac{3\lambda^2(\mu)}{(4\pi)^{d/2}\Gamma(d/2)} \,. \tag{18}$$

Fixed points occur when the so-called beta function vanishes, $\beta^{(\lambda)}(\lambda_*) = 0$. Considering $\lambda > 0$ (for a stable vacuum), such a fixed point can occur when $d < 4$,

$$\lambda_* = \frac{4-d}{3}(4\pi)^{d/2}\Gamma(d/2) \,. \tag{19}$$

When $\lambda > \lambda_*$, or $0 < \lambda < \lambda_*$ (provided higher order perturbations are sufficiently small), renormalization evoluation will cause $\lambda(\mu) \to lambda_*$ as μ decreases: λ_* is an IR fixed point.

Exercise: heavy fermion effective theory. We have considered integrating out some momentum modes of a field. Let us consider now integrating out the entire antiparticle component of a field. Consider the lagrangian for a massive fermion charged under an abelian gauge group:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \,, \tag{20}$$

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative.

a) Introduce the field redefinition

$$\psi(x) = e^{-imv\cdot x}[h(x) + H(x)] \,, \tag{21}$$

where v is a four-velocity ($v^2 = 1$), and h, H are defined by

$$h(x) = \frac{1}{2}(1+\not{v})e^{imv\cdot x}\psi(x) \,, \quad H(x) = \frac{1}{2}(1-\not{v})e^{imv\cdot x}\psi(x) \,. \tag{22}$$

Consider the $m \to \infty$ limit of (20). Show that H can be ignored in this limit, and find the leading term involving h.

b) Introduce the further field redefintion

$$h(x) = V(x)\tilde{h}(x) \,, \tag{23}$$

where $V(x)$ is a Wilson line along v^μ,

$$V(x) = P\left\{\exp\left[ig\int_{-\infty}^{0} ds\, v^\mu A_\mu(x+sv)\right]\right\} \,. \tag{24}$$

Suppose that the original field transforms as $\psi(x) \to e^{i\alpha(x)}\psi(x)$ under a gauge transformation, and that $\alpha(x)$ vanishes outside a finite region. How does $\tilde{h}$ transform? Find the form of the leading Lagrangian from (a) in terms of $\tilde{h}$.