1 Path integrals introduction

The basic object that we analyzed in perturbation theory was

$$\begin{aligned} G(x_1, x_2, \dots) &= \langle \mathrm{vac} | T\{\phi(x_1)\phi(x_2)\dots\} | \mathrm{vac}\rangle \\ &= \frac{\langle 0 | T\{\phi_I(x_1)\phi_I(x_2)\dots \exp\left[i\int d^dy \mathcal{L}_I[\phi_I(y)]\right]\} | 0\rangle}{\langle 0 | T\{\exp\left[i\int d^dy \mathcal{L}_I[\phi_I(y)]\right]\} | 0\rangle} \end{aligned} \tag{1}$$

The path integral approach employs the ansatz

$$G(x_1, x_2, \dots) = \frac{\int [d\phi]\phi(x_1)\phi(x_2)\dots \exp\left[i\int d^dy \mathcal{L}[\phi(y)]\right]}{\int [d\phi] \exp\left[i\int d^dy \mathcal{L}[\phi(y)]\right]} \,. \tag{2}$$

Our first tasks will be to define the formal integration measure

$$[d\phi] \sim \prod_x d\phi(x) \,, \tag{3}$$

and to introduce appropriate boundary conditions for which we can justify the identification (1) and (2). We will then proceed to define gauge theories based on the path integral approach. This lecture reviews two regularizations of the path integral, first as a discretization in timeslices, then as an expansion in basis functions.

1.1 $0+1$-dimensional example

To make contact with familiar quantum mechanical normalizations it is useful to consider the case of a scalar field in $0+1$ dimensions, with the field $x(t)$ representing, e.g., the position of a structureless particle.

1.1.1 Hilbert space

Thus, consider a quantum mechanical system described by canonical coordinate x and conjugate momentum p. Viewed as operators on Hilbert space,

$$[\hat{x}, \hat{p}] = i \,, \quad [\hat{x}, \hat{x}] = 0 \,, \quad [\hat{p}, \hat{p}] = 0 \,, \tag{4}$$

and the Heisenberg picture operators

$$\hat{x}(t) = e^{i\hat{H}t}\hat{x}e^{-i\hat{H}t} \,, \quad \hat{p}(t) = e^{i\hat{H}t}\hat{p}e^{-i\hat{H}t} \,, \tag{5}$$

obey[1]

$$\begin{aligned} \dot{\hat{x}} &= \frac{\partial H}{\partial p} \,, \\ \dot{\hat{p}} &= -\frac{\partial H}{\partial x} \,. \end{aligned} \tag{6}$$

We may as usual introduce a basis of x eigenstates,

$$\hat{x}|x\rangle = x|x\rangle \,, \quad \mathbb{1} = \int_{-\infty}^{\infty} dx |x\rangle\langle x| \,, \quad \langle x'|x\rangle = \delta(x' - x) \,, \tag{7}$$

or a basis of p eignenstates,

$$\hat{p}|p\rangle = p|p\rangle \,, \quad \mathbb{1} = \int_{-\infty}^{\infty} dp |p\rangle\langle p| \,, \quad \langle p'|p\rangle = \delta(p' - p) \,. \tag{8}$$

[1]For simplicity, and since most of the field theories we will consider have this property, we suppose that $\partial H/\partial p$ and $\partial H/\partial x$ do not suffer ordering ambiguities. The general case can be treated in detail.

Using the commutation relations, we have for any number a,

$$\hat{x}\left(e^{i\hat{p}a}|x\rangle\right) = e^{i\hat{p}a}(\hat{x}-a)|x\rangle = (x-a)\left(e^{i\hat{p}a}|x\rangle\right)\,, \tag{9}$$

so that we may identify

$$e^{i\hat{p}a}|x\rangle = |x-a\rangle\,. \tag{10}$$

It follows that

$$\langle x|p\rangle = \frac{e^{ipx}}{\sqrt{2\pi}}\,. \tag{11}$$

We define basis states having definite position or momentum at time t as

$$\hat{x}(t)|x;t\rangle = x|x;t\rangle\,, \quad \hat{p}(t)|p;t\rangle = p|p;t\rangle\,. \tag{12}$$

The quantity $\langle x';t'|x;t\rangle$ is thus identified as the amplitude to find a particle at position x' at time t' if it was at position x at time t.

Exercise: Show that $|x;T\rangle$ is the Schrodinger picture state at time $t=-T$ corresponding to the Heisenberg state $|x\rangle$ at time $t=0$. Show the relations

$$\langle x';t|x;t\rangle = \delta(x'-x)\,, \quad \langle p';t|p;t\rangle = \delta(p'-p)\,, \quad \int dx|x;t\rangle\langle x;t| = \mathbb{1}\,, \quad \int dp|p;t\rangle\langle p;t| = \mathbb{1}\,, \quad \langle x;t|p;t\rangle = \frac{e^{ip\cdot x}}{\sqrt{2\pi}}\,. \tag{13}$$

1.1.2 Time slices

As an exact relation,

$$\langle x';t+dt|x;t\rangle = \langle x';t|e^{-i\hat{H}dt}|x;t\rangle = \int dp\langle x';t|e^{-i\hat{H}dt}|p;t\rangle\langle p;t|x;t\rangle\,. \tag{14}$$

We notice that

$$\hat{H}(\hat{x},\hat{p}) = \hat{H}(\hat{x}(t),\hat{p}(t))\,, \tag{15}$$

and we assume that $\hat{H}$ is expressed with all $\hat{x}$'s to the left and all $\hat{p}$'s to the right. Then to first order in dt,

$$\langle x';t|e^{-i\hat{H}dt}|p;t\rangle = e^{-iH(x',p)dt}\langle x';t|p;t\rangle + \mathcal{O}(dt^2)\,,\,. \tag{16}$$

and

$$\langle x';t+dt|x;t\rangle == \int\frac{dp}{2\pi}e^{ip(x'-x)-iH(x',p)dt} + \mathcal{O}(dt^2)\,.\,. \tag{17}$$

For a finite time interval, let us introduce N time slices,

$$\langle x';t'|x;t\rangle = \int dx_1\dots dx_N\langle x';t'|x_N;t_N\rangle\langle x_N;t_N|x_{N-1};t_{N-1}\rangle\dots\langle x_1;t_1|x;t\rangle\,, \tag{18}$$

identifying

$$t'-t = (N+1)dt\,, \quad x_0 = x\,, \quad x_{N+1} = x'\,. \tag{19}$$

Employing (17) at each timeslice, we then have

$$\langle x';t'|x;t\rangle = \int\prod_{i=1}^{N}dx_i\prod_{j=0}^{N}\frac{dp_j}{2\pi}\exp\left[\sum_{k=1}^{N+1}ip(x_k-x_{k=1})-iH(x_k,p_k)dt\right] + \mathcal{O}(Ndt^2)\,. \tag{20}$$

With $dt\sim 1/N$, the result becomes exact as $N\to\infty$.

1.1.3 Lagrangian formulation

The field theories of interest will generally have the form (extended from $0+1$ to d dimensions, $x \to \phi(\boldsymbol{x})$)

$$H = \frac{1}{2}p^2 + V(x)\,, \tag{21}$$

so that each p_j integration is of the form

$$\int \frac{dp_j}{2\pi} \exp\left[i\left(-\frac{1}{2}p_j^2 dt + p_j(x_{j+1}-x_j)\right)\right] = \sqrt{\frac{1}{2\pi i dt}} \exp\left[\frac{i(x_{j+1}-x_j)^2}{2dt}\right], \tag{22}$$

where we complete the square and perform the remaining gaussian integral. Using this result for each time slice results in

$$\langle x'; t'|x; t\rangle = \lim_{N\to\infty} \left(\frac{1}{2\pi i dt}\right)^{\frac{N+1}{2}} \int \prod_{i=1}^{N} dx_i \exp\left[i \sum_{j=0}^{N} \left(\frac{(x_{j+1}-x_j)^2}{2dt^2} - V(x_j)\right) dt\right]. \tag{23}$$

In the extenstion to quantum field theory we will not bother keeping track of the (divergent as $dt \to 0$) normalization factor.[2]

1.1.4 Continuum limit

In the limit of large N we formally write

$$\langle x'; t'|x; t\rangle = \int_{\substack{x(t)=x \\ x(t')=x'}} \prod_t dx(t) \exp\left[i \int_t^{t'} dt \left(\frac{1}{2}\dot{x}^2 - V(x)\right) dt\right], \tag{24}$$

where the quantity being integrated in the exponent may be identified with the lagrangian,

$$\frac{1}{2}\dot{x}^2 - V(x) = L\,. \tag{25}$$

The "regularization" (23) is one way to interpret (24) (and in fact is the method by which we derived (24) starting from canonical operators in quantum mechanical Hilbert space). Another regularization is obtained by expanding in a basis of functions on the chosen time interval, taking the expansion coefficients as integration variables. Consider for definiteness the harmonic oscillator problem,

$$V(x) = \frac{1}{2}\omega^2 x^2\,, \tag{26}$$

on the time interval $0 < t < T$, with $x(0) = x(T) = 0$. We may identify

$$\int_0^T dt \left(\frac{1}{2}\dot{x}^2 - V(x)\right) \equiv \frac{1}{2}\langle x|A|x\rangle\,, \tag{27}$$

where

$$A = -\left(\frac{d}{dt}\right)^2 - \omega^2\,. \tag{28}$$

Consider the orthonormal basis of functions satisfying the chosen boundary conditions,

$$x_n(t) = \sqrt{\frac{2}{T}} \sin\frac{\pi n t}{T}\,, \quad A x_n = \lambda_n x_n\,, \quad \lambda_n = \frac{\pi^2 n^2}{T^2} - \omega^2\,. \tag{29}$$

Thus

$$x(t) = \sum_{n=1}^{\infty} c_n x_n(t) \implies \frac{1}{2}\langle x|A|x\rangle = \frac{1}{2}\sum_n c_n^2 \lambda_n\,. \tag{30}$$

[2]This factor depends on the regularization scheme, which must be specified. For many quantum mechanical problems (e.g., the Schrodinger Coulomb system) most observables are independent of UV regulator, and it makes sense to retain a rigid connection to the normalizations of the familiar Hilbert space formulation as in (23).

With the identification

$$[dx] = \prod_t dx(t) \propto \prod_n \frac{dc_n}{\sqrt{2\pi}} \equiv [dc]\,, \tag{31}$$

we have

$$\int [dx] \exp\left[-\frac{i}{2}\langle x|A|x\rangle\right] \propto \prod_n \int \frac{dc_n}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(i\lambda_n)c_n^2\right] = \prod_n (i\lambda_n)^{-\frac{1}{2}} = (\det iA)^{-\frac{1}{2}}\,. \tag{32}$$

The $\propto$ in (32) implies that we have not determined the overall normalization of the measure to rigidly connect with the canonical quantum mechanics normalization. We may however compute quantities for which this normalization drops out. For example, consider,

$$\begin{aligned}
\frac{\int [dx] x(t_1)x(t_2) \exp\left[-\frac{i}{2}\langle x|A|x\rangle\right]}{\int [dx] \exp\left[-\frac{i}{2}\langle x|A|x\rangle\right]} &= \sum_{n,m} x_n(t_1)x_m(t_2) \frac{\int [dc] c_n c_m \exp[\left[-\frac{i}{2}\langle x|A|x\rangle\right]}{\int [dc] \exp[\left[-\frac{i}{2}\langle x|A|x\rangle\right]} \\
&= \sum_{n,m} x_n(t_1)x_m(t_2)\delta_{nm} \frac{\partial}{\partial i\lambda_n} \log\det(iA) \\
&= \sum_n \frac{x_n(t_1)x_n(t_2)}{i\lambda_n}
\end{aligned} \tag{33}$$

Consider for example,

$$t_1 = \frac{T}{2} + \tau\,, \quad t_2 = \frac{T}{2}\,, \quad \tau/T \to 0\,. \tag{34}$$

In this limit, we have ($\alpha = \omega T$)

$$\sum_{n=1}^{\infty} \frac{x_n(t_1)x_n(t_2)}{i\lambda_n} = \frac{1}{\pi\omega} \sum_{n=1}^{\infty} \frac{\alpha}{\alpha^2 + n^2} [1 - (-1)^n] \to \frac{1}{2\omega}\,. \tag{35}$$

We may identify this as the free particle correlation function in $0+1$ dimensions,

$$\langle 0|T\{x(\tau)x(0)\}0\rangle = \frac{\int dp_0}{2\pi} \frac{e^{-ip_0 t}}{p_0^2 - \omega^2 + i0} = \frac{1}{2\omega}\,. \tag{36}$$

Exercise: Compute $\langle x'; t'|x; t\rangle$ from (24) for the case $x = 0$, $t = 0$, and general $x' = X$, $t' = T$. Hint: Write $x(t)$ as the sum of a classical solution and perturbation,

$$x(t) = x_c(t) + \sum_{n=1}^{\infty} c_n x_n(t)\,, \tag{37}$$

where

$$x_c(t) = X \frac{\sin\omega t}{\sin\omega T}\,. \tag{38}$$

By examining the free ($\omega = 0$) case you should find that the overall normalization consistent with (7) follows from

$$[dx] = \left(\frac{1}{2\pi i T}\right)^{\frac{1}{2}} \prod_n \left(\frac{1}{2\pi i T}\right)^{\frac{1}{2}} \frac{n\pi}{\sqrt{T}} dc_n\,. \tag{39}$$

The same result may be obtained from (23) by evaluating the N-dimensional gaussian integral, then taking $N \to \infty$.