1 Recap of scalar, spinor and vector fields

From last quarter, recall that we have constructed scalar, spinor and vector fields. To avoid complications associated with gauge fields, consider first the case of massive vector fields.

1.1 Real scalar

From the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \dots \tag{1}$$

we introduce the interaction picture field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(e^{-ip\cdot x}a_{\boldsymbol{p}} + e^{ip\cdot x}a^\dagger_{\boldsymbol{p}}\right)\bigg|_{p^0=E_{\boldsymbol{p}}}\,, \tag{2}$$

with $E_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2}$ and annihilation and creation operators obeying

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}] = 0\,, \quad [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = 0\,. \tag{3}$$

Exercise: Show that (2) satisfies the equation of motion derived from (1). Derive the conjugate momentum $\pi(\boldsymbol{x},t)$ and show that at equal times $[\phi(\boldsymbol{x},t), \pi(\boldsymbol{y}',t)] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$.

1.2 Complex scalar

For two real scalars with equal mass,

$$\mathcal{L} = \sum_{i=1}^{2}\frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{1}{2}m^2(\phi_i)^2 + \dots \tag{4}$$

we may expand each ϕ_i as in (2). Introducing the complex combinations

$$\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \quad \Phi^*(x) = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{5}$$

we may write

$$\mathcal{L} = (\partial_\mu\Phi)^*(\partial^\mu\Phi) - m^2\Phi^*\Phi + \dots \tag{6}$$

and expand the interaction picture field

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(e^{-ip\cdot x}a_{\boldsymbol{p}} + e^{ip\cdot x}b^\dagger_{\boldsymbol{p}}\right)\bigg|_{p^0=E_{\boldsymbol{p}}}\,, \tag{7}$$

where $a_{\boldsymbol{p}} = (a_{1,\boldsymbol{p}} + ia_{2,\boldsymbol{p}})/\sqrt{2}$, $b_{\boldsymbol{p}} = (a_{1,\boldsymbol{p}} - ia_{2,\boldsymbol{p}})/\sqrt{2}$,

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \quad [b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{8}$$

and all other commutators vanish.

Exercise: Show that (7) satisfies the equation of motion derived from (6). Derive the conjugate momentum $\Pi(\boldsymbol{x},t)$ and show that at equal times $[\Phi(\boldsymbol{x},t), \Pi(\boldsymbol{y}',t)] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$.

1.3 Weyl fermion

The lagrangian for a fermion field transforming irreducibly under Lorentz transformations is

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{m}{2}(\chi^T i\sigma^2\chi - \chi^\dagger i\sigma^2\chi^*) + \dots\,. \tag{9}$$

The field in the interaction picture may be expanded

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(e^{-ip\cdot x}\sum_{s=1}^{2} a_{\boldsymbol{p},s}\sqrt{p\cdot\sigma}\xi_s + e^{ip\cdot x}\sum_{s=1}^{2} a^\dagger_{\boldsymbol{p},s}\sqrt{p\cdot\sigma}(-i\sigma^2\xi_s^*)\right)\Bigg|_{p^0=E_{\boldsymbol{p}}}, \tag{10}$$

where ξ_s are a basis of two-component spinor wavefunctions, and the annihilation and creation operators satisfy anticommutation relations,

$$\{a_{\boldsymbol{p},s}, a^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,,\quad \{a_{\boldsymbol{p},s}, a_{\boldsymbol{p}',s'}\} = 0\,,\quad \{a^\dagger_{\boldsymbol{p},s}, a^\dagger_{\boldsymbol{p}',s'}\} = 0\,. \tag{11}$$

Here $\sigma^\mu = (1,\boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1,-\boldsymbol{\sigma})$.

Exercise: Show that (10) satisfies the equation of motion derived from (9). Derive the conjugate momentum $\pi(\boldsymbol{x},t)$ and show that at equal times $\{\chi(\boldsymbol{x},t),\pi(\boldsymbol{y}',t)\} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$.

1.4 Dirac fermion

Consider two Weyl fermions of equal mass,

$$\mathcal{L} = \sum_{i=1}^{2}\chi^{(i)\dagger} i\bar{\sigma}\cdot\partial\chi^{(i)} + \frac{m}{2}(\chi^{(i)T} i\sigma^2\chi^{(i)} - \chi^{(i)\dagger} i\sigma^2\chi^{(i)*})\,. \tag{12}$$

and define the linear combinations,

$$\psi_L = \frac{1}{\sqrt{2}}(\chi^{(1)} + i\chi^{(2)})\,,\quad \psi'_L = \frac{1}{\sqrt{2}}(\chi^{(1)} - i\chi^{(2)})\,. \tag{13}$$

If we further introduce

$$\psi_R = i\sigma\psi_L'^*\,, \tag{14}$$

and collect ψ_L and ψ_R into

$$\Psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}, \tag{15}$$

the lagrangian (12) becomes

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi\,, \tag{16}$$

where $p\!\!\!/ = \gamma^\mu p_\mu$, in the basis of Dirac matrices

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}, \tag{17}$$

Exercise: Verify that (16) is equivalent to (12).

Finally, we may expand

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(e^{-ip\cdot x}\sum_{s=1}^{2} a_{\boldsymbol{p},s}u_s(\boldsymbol{p}) + e^{ip\cdot x}\sum_{s=1}^{2} b^\dagger_{\boldsymbol{p},s}v_s(\boldsymbol{p})\right)\Bigg|_{p^0=E_{\boldsymbol{p}}}, \tag{18}$$

where $a_{\boldsymbol{p},s} = (a^{(1)}_{\boldsymbol{p},s} + ia^{(2)}_{\boldsymbol{p},s})/\sqrt{2}$, $b_{\boldsymbol{p},s} = (a^{(1)}_{\boldsymbol{p},s} - ia^{(2)}_{\boldsymbol{p},s})/\sqrt{2}$, and

$$u_s(\boldsymbol{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\xi_s\end{pmatrix},\quad v_s(\boldsymbol{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}(-i\sigma^2\xi_s^*)\\ -\sqrt{p\cdot\bar{\sigma}}(-i\sigma^2\xi_s^*)\end{pmatrix}. \tag{19}$$

The annihilation and creation operators satisfy anticommutation relations,

$$\{a_{\boldsymbol{p},s}, a^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \quad \{b_{\boldsymbol{p},s}, b^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{20}$$

with all other anticommutators vanishing.

Exercise: Show that (18) satisfies the equation of motion derived from (16). Derive the conjugate momentum $\Pi(\boldsymbol{x},t)$ and show that at equal times $\{\Psi(\boldsymbol{x},t), \Pi(\boldsymbol{y}',t)\} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$.

1.5 (Massive) vector

From the lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{\lambda^2}{2}A_\mu A^\mu\,, \tag{21}$$

we introduce the interaction picture field,

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}}\left(e^{-ip\cdot x}\sum_{s=1}^{3} a_{\boldsymbol{p},s}\epsilon^\mu_s(\boldsymbol{p}) + e^{ip\cdot x}\sum_{s=1}^{3} a^\dagger_{\boldsymbol{p},s}\epsilon^\mu_s(\boldsymbol{p})^*\right)\Bigg|_{p^0=E_{\boldsymbol{p}}}\,, \tag{22}$$

2 Perturbation theory

Using LSZ reduction, physical S-matrix elements may be extracted from the correlation functions (here focusing on the real scalar field case),

$$\langle \text{vac}|T\{\phi(x_1)\phi(x_2)\ldots\}|\text{vac}\rangle\,. \tag{23}$$

An essential formula for perturbative quantum field theory is the relation of the above correlators to an expansion involving interaction picture fields,

$$\langle \text{vac}|T\{\phi(x_1)\phi(x_2)\ldots\}|\text{vac}\rangle = \frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\ldots\exp\left[i\int d^dx\mathcal{L}_I(x)\right]\}|0\rangle}{\langle 0|T\{\exp\left[i\int d^dx\mathcal{L}_I(x)\right]\}|0\rangle}\,, \tag{24}$$

where $|0\rangle$ is the free vacuum, $\phi_I(x)$ is the interaction picture field expressed in terms of creation and annihilation operators ((2) or in general also (7), (10), (18)) and $\mathcal{L}_I(x)$ is the interaction lagrangian. In the absence of constraints (e.g., for the massive vector case where A^0 acts as an auxiliarly field) $\mathcal{L}_I$ is the same function of $\phi_I(x)$ as the full lagrangian is of $\phi(x)$.

The relation (24) may be expanded in powers of small couplings appearing in $\mathcal{L}_I$. Upon (anti-) commuting all $a_{\boldsymbol{p}}$'s to the right and all $a^\dagger_{\boldsymbol{p}}$'s to the left, the result is the sum of contractions (since $a_{\boldsymbol{p}}|0\rangle = 0$) formalized as Wick's theorem,

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\ldots\}|0\rangle = \left[\,\overbrace{\phi(x_1)\phi}(x_2)\,\overbrace{\phi(x_3)\phi}(x_4)\ldots\right] + \ldots\,, \tag{25}$$

where

$$\overbrace{\phi(x_1)\phi}(x_2) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle \equiv D_F(x_1,x_2)\,. \tag{26}$$

The correlation functions (23) are thus reduced to integrals over sums of products of $D_F(x,y)$. The various contractions are conveniently expressed as Feynman rules. It is often easiest to go back to the starting point (24) when working out combinatorial factors.